ROOT LOCUS METHOD FOR SOLUTION OF HIGH ORDER ALGEBRAIC EQUATIONS

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The root locus diagram used in control engineering is a useful tool for mathematicians and engineers to solve algebraic equations.

In mathematics there are general methods for the solution of algebraic equations and others for special types. All these methods become laborious and too long to be practical in the case of high order equations (higher than second, third or fourth order).

The solution of these equations using the root locus method is simply by factorizing the equation into two parts, each part containing several factors. This is much easier than factorizing into one part which is the normal way for the solution of algebraic equations.

Examples .

$$x^6 + x^5 + 2x^3 + x - 1 = 0 \qquad (1)$$

This may be divided into two parts which can be factorized in several ways.

1.
$$(x^6-1)+(x^5+2x^3+x)=0$$
$$(x^3-1)(x^3+1)+x(x^4+2x^2+1)=0$$
$$(x-1)(x+1)(x^2+x+1)(x^2-x+1)+x(x^2+1)^2=0$$

2.
$$(x^6+2x^3-1)+(x^5+x)=0$$
$$(x^3+1+\sqrt{2})(x^3+1-\sqrt{2})+x(x^4+1)=0$$
$$(x+\sqrt[3]{1+\sqrt{2}})\left[x^2-\sqrt[3]{1+\sqrt{2}}\,x+\sqrt[3]{(1+\sqrt{2})^2}\right](x+\sqrt[3]{1-\sqrt{2}})\left[x^2+\sqrt[3]{1-\sqrt{2}}\,x+\sqrt[3]{(1-\sqrt{2})^2}\right]+x(x^2-j)(x^2+j)=0$$

3.
$$(x^6-x^3-2)+(x^5+x^3+x)=0$$
$$(x^3-2)(x^3+1)+x(x^4+x^2+1)=0$$
$$(x-\sqrt[3]{2})(x^2+\sqrt[3]{2}\,x+\sqrt[3]{4})(x+1)(x^2-x+1)+x(x^2+2-\sqrt{3})(x^2+2+\sqrt{3})=0$$

From the above examples it is seen that it is much easier to factorize an equation into two parts than in one part since there are more possibilities to do so in this case than in the ordinary factorizing.

After the equation has been factorized as shown above it is written in the form

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$$(x-p_1)(x-p_2)(x-p_3)(x-\cdots) \quad + K(x-z_1)(x-z_2)(x-z_3)(x-\cdots)\ldots = 0 \qquad (2a)$$

which may be rewritten as follows :

$$1 + K\frac{(x-z_1)(x-z_2)(x-z_3)(\cdots)(x-z_i)}{(x-p_1)(x-p_2)(x-p_3)(\cdots)(x-p_K)} = 0 \qquad ----(2b)$$

This equation can be drawn on the complex plane with K as variable.

Root Locus Technique:

The root locus diagram in control engineering is a plot of the roots of the charcteristic equation of the closed loop system as a function of the gain of the loop transfer function.

Consider the equation (2b) which may be considered as a closed loop equation with K as the gain of the loop transfer function.

Now we mean by the root locus for this equation, all the points on the complex plane which satisfies the equation. The points on the complex plane whose coordinates are Z_1, Z_2, Z_3 are calleed the zeros of the equation and the points P_1, P_2, P_3 are called the poles of the equation.

There are some properties of the root locus diagram which are useful in plotting it. These are:

1 The loci start from the poles and terminate at zeros.

2. The root loci are symmetrical about the real axis.

3 The number of seperate loci equals the number of poles or zeros whichever number is larger.

4 The loci near infinity for large values of K approach asymptotic lines whose directions are given by the angles

$$\theta_i = \pm\frac{n\pi}{P-Z} \qquad \text{where } n \text{ is an odd integer} \qquad -----(3)$$

P − Z is the difference between number of finite poles and zeros.

5 The asymptotes intersect on the real axis at σ_1, which is determined by the formula

$$\sigma_1 = \frac{\sum \text{poles} - \sum \text{zeros}}{P-Z} \qquad ------------(4)$$

6 The parts of the real axis which comprise sections of the loci are to be the left of an odd number of critical frequencies (poles or zeros).

7 The angle of departure from a pole P_x is given by

$$\phi_p = \sum_{j=1} \arg(P_x - z_j) - \sum_{\substack{i=1 \\ i\neq x}} \arg(P_x - P_i) \mp 180^\circ \qquad ---------(5)$$

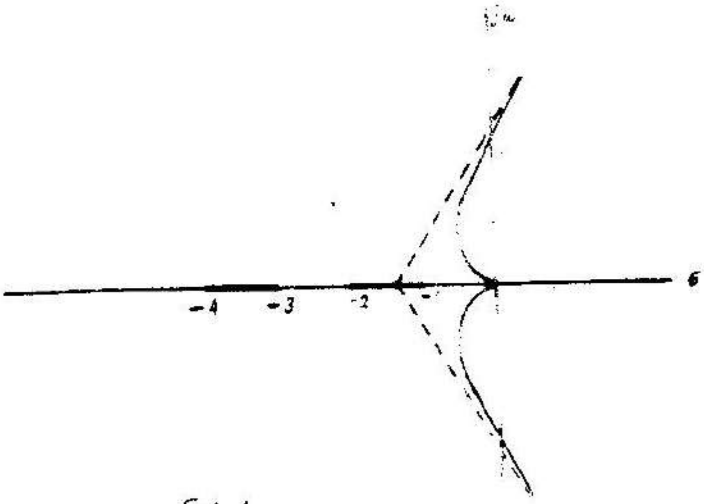

Fig 1

Solution of Algebraic Equations Using Root Locus Technique

If the root locus is plot for a certain equation, then each point on the locus is corresponding to a certain value of K. Then on each locus there is a point which has a value of $K = K'$. These points are the solution of the algebraic equation which has the gain constant $K = K'$.

Example: Assume an equation of the seventeenth order has been factorized into the form:

$$X^5(X-3)^2(X^3-1)^2(X+2)^4 + (X^3+1)^2(X-2)^2 = 0 \quad \text{.......... (9a)}$$

It can be written in the form

$$1 + K\frac{(x^3+1)^2(x-2)^2}{x^5(x-3)^2(x^3-1)^2(x+2)^4} = 0 \quad - - - ——— - \quad (9b)$$

where $K = 1$

The root locus for this equation may be drawn as follows:

1. The zeros are: $x = 2, 2, -1, -1, \frac{1}{2}+j\frac{\sqrt{3}}{2}, \frac{1}{2}+j\frac{\sqrt{3}}{2}, \frac{1}{2}-j\frac{\sqrt{3}}{2}$ and $\frac{1}{2}-j\frac{\sqrt{3}}{2}$

2. The poles are: $x = 0, 0, 0, 0, 0, 3, 3, 1, 1, -\frac{1}{2}+j\frac{\sqrt{3}}{2}, -\frac{1}{2}+j\frac{\sqrt{3}}{2}, -\frac{1}{2}-j\frac{\sqrt{3}}{2}, -\frac{1}{2}-j\frac{\sqrt{3}}{2}, -2, -2, -2, -2.$

3. There are root loci on the real axis between 0 and ... and −2 and to the left of −2 only.

4. The point where the asymptotes intersect the real axis is given by equation (4) which will give

8. Similarly the angle of departure from a zero Z_x :

$$\phi_z = \sum_{i=1} \arg (Z_x - p_i) - \sum_{\substack{j=1 \\ j \neq x}} \arg (Z_x - Z_j) \pm 180^\circ \qquad \text{— — — —(6)}$$

9. The breakaway points (points where the loci breaks sharply into new loci) are determined by solving the equation

$$\frac{dK}{ds} = \frac{d}{ds}\left(-\frac{1}{G(s)}\right) = 0 \qquad \text{— — — — — — — — — (7)}$$

10. The intersection of the loci with imaginary axis is given by the Routh-Hurwitsz test which is beyond the scope of this paper.

The following example will explain the plot of a fifth order system.

Let us consider the transfer function of this system to be

$$G(s) = \frac{K(s+1)(s+2)}{s^3(s+3)(s+4)} \qquad \text{— — — — — — — — — (8)}$$

The following steps will be followed in plotting it according to the properties discussed above:-

1. The loci starts from the poles

   $s = 0, 0, 0, -3$ and -4

   and terminates at the zeros

   $Z = -1$ and -2.

2. The number of separate loci $= 5$

3. The asymptotes make angles of

   $$\theta = \pm\frac{180^\circ}{5-2} = \pm 60^\circ \text{ and } 180^\circ$$

4. These asymptotes intersect the real axis at

   $$= \frac{-3-4+0+0+0-(-1-2)}{5-2} = -1.33$$

5. There is a locus on the real axis between (-1) and (0), between (-3) and (-2), and to the left of the point (-4)

6. The angle of departure from a pole at $s = 0$ is given by $\theta =$

   $0 + 0 - 0 - 0 - 0 \pm 180^\circ = 180^\circ$

No further steps are necessary since the shape of the loci is known now and will be as shown by figure (1).

$\sigma_1 = -0.73 + j0$

5. The angles of intersection are given by equation (8) which will give

$\theta = 20°, 60°, 100°, 140°, 180°, 220°, 260°, 300°$ and $340°$

6. The angle of departure (and termination) from poles and zeros may be calculated using equations (5) and (6). These will be

$\phi = \pm 90°$ at $p = 3$, $z = 2$, $p = 1$ and $p = -2$

$\phi = \pm 36°$ and $\pm 72°$ at $p = 0$ and so on

7. The points where K = 1 may be found such that the ratio of the product of the distances between such point with the zeros to those with the poles may be equal to K = 1 here

Since the solution is a graphical solution it is recommended to draw it using larger scale to get accurate results.

8. To be sure that the points to be tested are really situated on the locus the angles which they are making with the zeros and poles may be measured and checked whether the difference between them equals to odd multiples of 180° or not.

9. A scale of 1 cm = 1 was chosen (twice the one shown in the figure) to draw it, then about forty trials were made to obtain, all the points where K = 1 which are located on the locus. The coordinates of these points are:

$-0.56,\ -1.616,\ -2.13,\ 0.35 \pm j0.28,\ -0.65 \pm j0.9,\ 1 \pm 0.037,\ -2.02 \pm j0.37,$
$3 \pm j0.055,\ -0.36 \pm j0.9,\ -0.035 \pm j0.495$

These values are the solution of the original equation which has seventeen roots. The accuracy of these values is better than 95 % in most cases depending on the scale used, the number of trials done and on the approximation made.

Conclusion:

The method of solution of high order algebraic equation of the form of equation (3b) above, or of the form

$$-K = \frac{(x - z_1)(x - z_2)(x - z_3)\cdots}{(x - p_1)(x - p_2)(x - p_3)\cdots} \qquad (3c)$$

may be understood fully if it is imagined that the magnitude of the right hand side (magnitude of the numerator divided by that of the denominator) equals that of the left hand side = K, and the angle of the right hand side (angle of numerator minus that of the denominator) equals to that of the left and equals to (2n + 1) × 180° where n is an integer.

References:

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